7 percolation theory

Sunday, February 23, 2020 10:40 PM

Last time, showed that components are either $O(\log n)$ or $\Omega(n)$ in size,

Let's prove the uniqueness of the large component.

No other large components

Claim: For any $\varepsilon > 0$, $p = \frac{1+\varepsilon}{n}$, w.h.p. there is only one giant component in $G(n,p)$, all all other components have size $O(\log n)$.

proof. Suppose $G(n,p)$ has δ prob. of 2 distinct components K_1 and K_2 of size $\omega(\log n)$.

Let $A = \{1, 2, \ldots, \frac{\varepsilon n}{2}\}$.

Then $Prob(|K_1 \cap A| = \omega(\log n)$ and $|K_2 \cap A| \cap \omega(\log n)) \geq \frac{\delta}{2}$,

because we can imagine randomly permuting vertex labels, and

both K_1 and K_2 w.h.p. have $\frac{\varepsilon}{4}$ fraction of their nodes in A. (expected $\frac{\varepsilon}{4}$)

Thus, if we can show there exists only 1 component that intersects A in $\omega(\log n)$ vertices, we would be done.

Let $B = V - A$, $|B| = n(1 - \frac{\varepsilon n}{2})$.

B has at least 1 giant component C^*, $|C^*| = \omega(\log n)$.

Let $C_1, C_2, C_3, \ldots$ be $\omega(\log n)$ components within A.

$\forall i$, there are $\omega(n \log n)$ potential edges between C_i and C^*.

Thus, $Prob(C_i \text{ not connected to } C^*) \leq (1-p)^{\omega(n \log n)} = \frac{1}{n^{\omega(1)}}$.

By union bound, all C_i's are connected to C^* w.h.p.

Thus, only 1 component intersects A in $\omega(\log n)$ vertices.

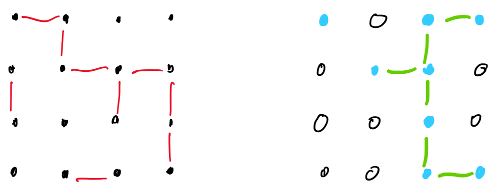
$\Rightarrow$ Only 1 large component in A.

Percolation theory

Percolation theory

We have spent basically all of our section on random graphs where each vertex was free to connect to some other vertex with prob. p.

But this is not very in the physical world, where there may be geometric constraints. Instead, let's consider electrical conductivity of a material.



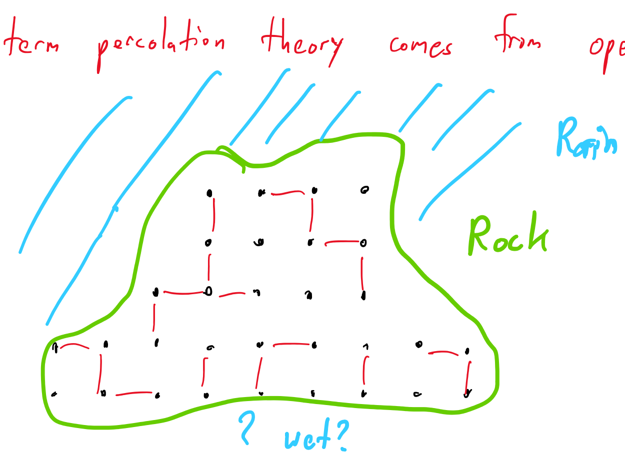
Bond percolation   Site percolation

Consider a square lattice $\mathbb{Z}^2$

Bond percolation: All vertices are present, but edges present w.p. p.

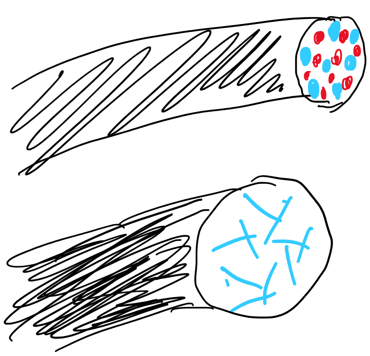
Site percolation: Vertices present w.p. p, all edges present b/t neighboring vertices.

The term percolation theory comes from open/closed channels for a fluid to flow.



Can approximate lattice as infinite if rock is much bigger than the channels of interest.

Tells us if a material is porous.



Composite wires can also be modelled using percolation theory to determine conductivity.

Many polymers are insulating, but you can get the strength of a polymer without needing it to be homogeneous, so you can instead dope it with a conductive filler.

Modelled with either site or bond percolation depending on features.
If they actually displace on a 1-1 basis, perhaps site-percolation.
If they are everywhere, but only sometimes link adjacent units, perhaps bond-percolation.

Basic questions:

Basic questions:

- Does there exist an infinite open cluster?
- What is the size distribution of open clusters?

Relationship to epidemic models on lattices, though many additional complications.

Def. Let $C(x)$ denote the component containing x in our random graph. (i.e. a random variable dependent on which edges are present)

Def. Let $C = C(0)$. Define $\theta(p) = \mathbb{P}_p(|C| = \infty)$. i.e. the probability that the origin is in an open component of infinite size.

Def. Let p_c be a constant, such that for $p < p_c$, $\theta(p) = 0$, and for $p > p_c$, $\theta(p) > 0$.

Aside: By Kolmogorov's 0-1 law (stating tail events have either 0 or 1 prob.), if $\theta(p) > 0$, then there exists almost surely an infinite open component.

Theorem: If $p < \frac{1}{3}$, $\theta(p) = 0$

proof. We will use the first moment method.

Let F_n be the event that there is a self-avoiding path of length n starting at 0 using only open edges.

For any given self-avoiding path in $\mathbb{Z}^2$, the prob. that all edges are open is p^n.

The total number of self avoiding paths of length n $\leq 4(3^{n-1})$.
(because after the first step, can't backtrack).

$\Rightarrow \mathbb{P}_p(F_n) \leq 4(3^{n-1})p^n \to 0$ as $n \to \infty$ since $p < \frac{1}{3}$.

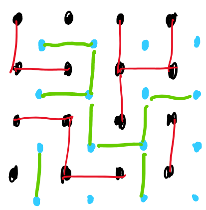
$\{|C| = \infty\} \subseteq F_n \;\; \forall n$. (because if you have an infinite component you must have a self-avoidant path of length n)

$\Rightarrow \mathbb{P}_p\{|C| = \infty\} = 0 \Rightarrow \theta(p) = 0$.

$\Rightarrow p_c \geq \frac{1}{3}$

Theorem (Harris, 1960): $\theta\left(\frac{1}{2}\right) = 0.$ $\left(p_c \geq \frac{1}{2}\right)$

For this proof, we will make use of the self-duality of $\mathbb{Z}^2$.



We can define a dual graph by translating down and to the right by $(0.5, 0.5)$, and defining and edge to be present precisely when the edge it would cross in the original graph is missing.

Note: If we had prob. p of a edge in the original graph, we have prob. $1-p$ of an edge in the dual graph.

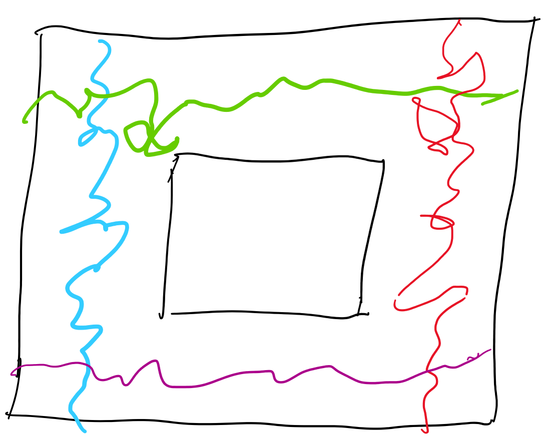
Suppose that $p < p_c$. Then $C(0)$ is finite almost surely.

$\Rightarrow$ there exists an open cycle in the dual graph encircling 0.

Conversely, if there is an open cycle in the dual graph encircling 0, then $C(0)$ is finite, because there are no open edges escaping the cycle.

So we just have to show the existence of an open cycle around 0 to prove that $\theta(p) = 0$, for any p.

Let $p = \frac{1}{2}$. Let's consider the prob. of an open cycle in an annulus composed of 4 separate open paths across rectangles.



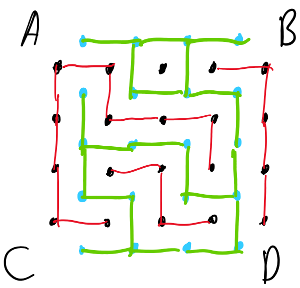
Lemma Let R be a rectangular $k \times \ell$ portion of the square lattice. Let R' be the corresponding rectangular $(k-1) \times (\ell+1)$ portion of the dual lattice.

Then there exists either an open horizontal path in R

Corresponding

Then there exists either an open horizontal path in R or an open vertical path in R'.

Argument is purely geometric.



Begin walk at upper left hand corner A and always keep the dual wall on the left and original wall on the right. Then, the walk must end at either B or C because the dual wall must be on the left and the orig wall on the right.

This walk is simultaneously a walk on both R and R'.
If it ends at B, then exists horizontal walk on R.
C, then exists vertical walk on R'.

Can't end in the middle because the walk continues in one of the graphs and tumbles back on the other graph on the other side. $\square$

Let $\mathbb{P}_p(H(R))$ be the prob. of a horizontal path in R
$\mathbb{P}_{1-p}(V(R'))$ be the prob. of a vertical path in R'.

$$\Rightarrow \quad \mathbb{P}_p(H(R)) + \mathbb{P}_{1-p}(V(R')) = 1$$

$$\Rightarrow \quad \mathbb{P}_{\frac{1}{2}}(H(R)) + \mathbb{P}_{\frac{1}{2}}(V(R')) = 1.$$

If R is an $(n+1)\times n$ rectangle, then R' is an $n\times(n+1)$ rectangle.

$$\Rightarrow \quad \mathbb{P}_{\frac{1}{2}}(H(R)) = \mathbb{P}_{\frac{1}{2}}(V(R')) = \frac{1}{2}.$$

Let S by an $n\times n$ square. The horizontal distance to travel is $< n+1$ (in R), so $\mathbb{P}_{\frac{1}{2}}(H(S)) \geq \frac{1}{2}$, $\forall n$.

We will now use a simplification of an argument by Russo, Seymour, Welsh (RSW theory) to prove there exists with positive prob. independent of n an open cycle in the annulus.

Russo, Seymour, Welsh (RSW theory) to prove there exists with positive prob. independent of n an open cycle in the annulus.

Thm RSW: For all k, there exists c_k so that for all n, we have that

$$\mathbb{P}_{\frac{1}{2}}\left(H_{kn,n}\right) \geq c_k ,$$

where $H_{kn \times n} = H(R)$ where R is a $kn \times n$ rectangle.

Definition: Let the state space $\Omega := \{0,1\}^J$. There is a partial order on Ω given by $\omega \preceq \omega'$ if $\omega_i \leq \omega_i'$ for all $i \in J$.

A function $f: \Omega \to \mathbb{R}$ is increasing if $\omega \preceq \omega'$ implies that $f(\omega) \leq f(\omega')$. An event is increasing if its indicator function is increasing.

Note: If J is a set of edges in a graph and x & y are vertices, then the event that there is an open path is an increasing event.

Thm 6.2: Let $X := \{X_i\}_{i \in J}$ be independent r.v. taking values 0 and 1. Let f and g be increasing functions. Then

$$\mathbb{E}\left(f(X)\, g(X)\right) \geq \mathbb{E} f(X) \cdot \mathbb{E} g(X).$$

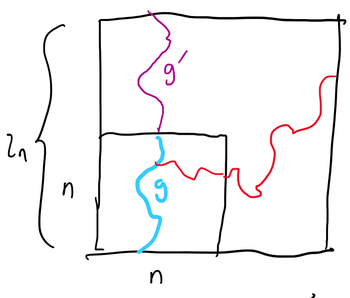
proof. See reference Steif, 2009.

Intuitively, while two events like existence of vertical and horizontal paths are not independent, they are positively correlated.

Proof by induction and direct computation of expectations.

proof of RSW: (not tight)

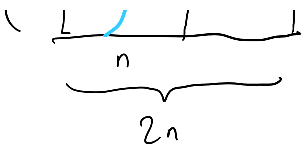
Let F_1 be the event that an open path connects the right side of a $2n \times 2n$ square with the top and bottom of the $n \times n$ square in the lower left quadrant.



$$\mathbb{P}_{\frac{1}{2}}\left(J_{2n,2n}\right) \geq \frac{1}{2} , \quad \mathbb{P}_{\frac{1}{2}}\left(J_{n,n}\right) \geq \frac{1}{2} .$$

Let g be an open vertical path in the lower left $n \times n$ square, and let g' be its reflection about $y = n$.

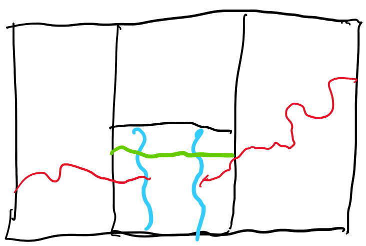
Let h be an open horizontal path in the $2n \times 2n$ square.



Let h be an open horizontal path in the $2n \times 2n$ square. By symmetry, it must cross either g or g', and has the same probability of either.

So $\mathbb{P}_{\frac{1}{2}}(F_1) \geq \frac{1}{2}\, \mathbb{P}_{\frac{1}{2}}(J_{2n,2n})\, \mathbb{P}_{\frac{1}{2}}(J_{n,n}) = 2^{-3}$. (Thm 6.2)

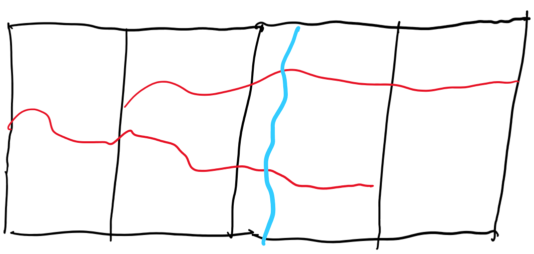
Let F_2 be the event that there is an open path across a $3n \times 2n$ rectangle.



We can break this up into two instances of F_1, coupled with a horizontal open path in the middle lower $n \times n$ square.

So $\mathbb{P}_{\frac{1}{2}}(F_2) \geq \left(\mathbb{P}_{\frac{1}{2}}(F_1)\right)^2 \cdot \mathbb{P}_{\frac{1}{2}}(J_{n\times n}) \geq 2^{-7}$.

Let F_3 be the event that there is an open path across a $4n \times 2n$ rectangle.



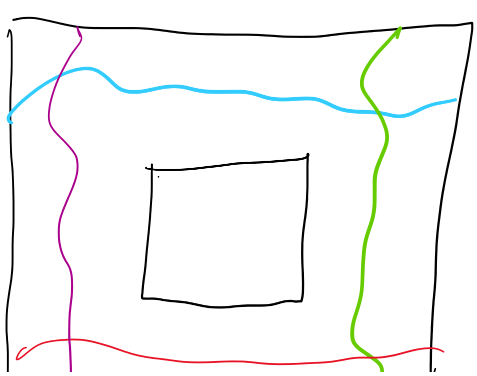
$\mathbb{P}_{\frac{1}{2}}(F_3) \geq \left(\mathbb{P}_{\frac{1}{2}}(F_2)\right)^2 \cdot \mathbb{P}_{\frac{1}{2}}(J_{2n\times 2n}) \geq 2^{-15}$.

Note that there is no dependence on n, so $\mathbb{P}_{\frac{1}{2}}(F_3) = \mathbb{P}_{\frac{1}{2}}(J_{4n\times 2n}) = \mathbb{P}_{\frac{1}{2}}(J_{2n\times n})$.

We can continue this process to get $\mathbb{P}_{\frac{1}{2}}(J_{kn\times n}) = c_k$ for some constant $c_k > 0$.

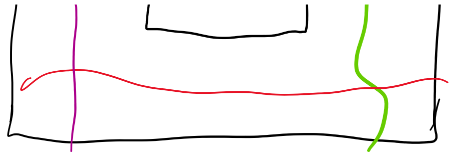
In particular $\mathbb{P}_{\frac{1}{2}}(J_{3n\times n}) \geq 2^{-31}$

Then, going back to the rectangular annulus, let F_A be the event that there is an open cycle in the $3n \times 3n$ annulus, not including the center $n \times n$ box.

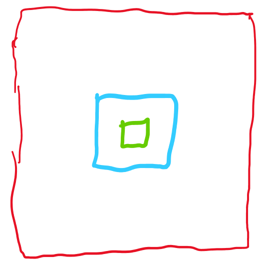


Then $\mathbb{P}(F_A) \geq (2^{-31})^4 = 2^{-124}$.

Note however, that this is independent of the size of the annulus, so we can surround with infinitely many larger annuli, each with ind. prob.



... infinitely many larger annuli, each with ind. prob. of having an open cycle.



$\mathbb{P}(\text{no open cycle in annulus}) \le 1 - 2^{-124}$

$\mathbb{P}(\text{no open cycle in } k \text{ non-overlapping annuli}) \le (1 - 2^{-124})^k$

As $k \to \infty$, this probability goes to 0, so almost surely, there is an open cycle in the dual graph surrounding the origin.

$$\Rightarrow \theta\left(\tfrac{1}{2}\right) = 0. \quad \Rightarrow \quad p_c \le \tfrac{1}{2}. \qquad \square$$

Thm: (Kesten, 1982) $p_c \ge \frac{1}{2}$.

Not proved here as a full proof would take too long.